RECURRENCE OF SUMS OF MULTIPLE MARKOV SEQUENCES

BY

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ABSTRACT

A d-dimensional random walk on a lattice is studied in which each step is bounded, and may depend on the previous m steps. It is proved that if trivial cases are excluded, there are no recurrent points for $d \geqq 3$, and conditions are given for the existence of sets, recurrent conditional on the first m steps, for $d = 1, 2$.

Let $X_1, X_2, \cdots$, be a sequence of d-dimensional random vectors with integer valued components, and let $S_n = X_1 + \cdots + X_n$. In the case where the X_i are mutually independent, each taking each of the values $\pm e_j$ with probability $1/(2d)$, where e_j is the vector with unity in its j-th component and zeros elsewhere, Pólya proved in [8] that

$$\Pr\{S_n = 0 \text{ for an infinity of values of } n\}$$

$$= 1, \text{ if } d = 1,2, \tag{1a}$$

$$= 0, \text{ if } d \geqq 3. \tag{1b}$$

The recurrence problem when the X_i form a Markov chain has been studied in generality only by Gillis [5], although other writers have dealt with the distribution of S_n in this case for $d = 1$, and Seth [9] has obtained some further results on recurrence for this value of d. Gillis proved that Pólya's result still holds if the X_i have the same range as above and satisfy symmetry conditions of the form

$$\Pr\{X_n = \xi \mid X_{n-1} = \xi'\} = \Pr\{X_n = -\xi \mid X_{n-1} = -\xi'\}, \tag{2}$$

and simplifying restrictions which he conjectured to be inessential, although his proof of (1b) did not cover the odd integers $\geqq 3$.

In this note I suppose that the X_i form a multiple Markov chain of arbitrary dependence, and are bounded, and by means of a different method based on the recurrence properties of finite Markov chains and generalisations of Pólya's result, [1,4], give a necessary and sufficient condition for (1) to hold essentially. (Complete enumeration of the particular cases corresponding to degeneracies in the transition structure would be tedious). However, unlike Gillis's analysis of the

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backward equations of the S_n process, this method does not readily provide information on the distribution of S_n.

Let us suppose that the X_i form a multiple Markov sequence of order m, [2, pp. 89, 185], subject to the condition that if $X_i^{(j)}$ is the jth component of X_i then with probability one

$$\| X_i \| = \sum_{j=1}^{d} |X_i^{(j)}| \leqq B. \tag{3}$$

Let $Z_i = (Z_i^{(1)}, \cdots, Z_i^{(m)}) = (X_i, \cdots, X_{i-m+1})$ be defined for $i \geqq m$ as the m dimensional vector whose components are consecutive vectors of the sequence $\{X_i\}$. The sequence $\{Z_i\}$ is a simple Markov chain M, [2], and by (3) it has a finite state space C. To avoid the uninteresting complications referred to above it will be assumed that M consists of a single ergodic class, (assumption E). Now choose an arbitrary state $\zeta_0 \epsilon C$, and denote by $T_1, T_2, \cdots$, the random values of n for which $Z_n = \zeta_0$ and define the following random variables,

$$U_i = T_{i+1} - T_i, \ V_i = S_{T_{i+1}} - S_{T_i}, \ W_k = \sum_{i=1}^{k} V_i$$

$$\chi_t(\zeta) = 1 \text{ if } Z_t = \zeta, = 0 \text{ otherwise}, \tag{4}$$

$$v_i(\zeta) = \sum_{t=T_i+1}^{T_{i+1}} \chi_t(\zeta).$$

It is known from recurrence theory, [3, Chapter 15, in particular Exercises 19,25] that the U_i are independent, and for $i \geqq 2$ are identically distributed with $\Pr\{U_i = k\} = a_k$ where for some positive $\lambda < 1$, $a_k \leqq \lambda^k$. The V_i are also independent and for $i \geqq 2$ are identically distributed. (This approach and similar notation has been used by Katz and Thomasian [7] to obtain probability bounds for sums of functions defined on Markov chains.) Further, since M has a finite state space it admits a stationary distribution $\beta(\zeta)$ with the property that for some constant A,

$$\mathscr{E} v_i(\zeta) = A\beta(\zeta), \ (= v(\zeta)).$$

Let $\alpha(\xi)$ be the marginal distribution induced by β on the first component of ζ.

LEMMA 1. *If conditions* (3) *and* (E) *hold, then for the sequence V_i defined above,*

$$\mathscr{E} V_i = A \sum_{\xi} \xi \alpha(\xi), \tag{i}$$

$$\mathscr{E} \| V_i \|^2 < \infty. \tag{ii}$$

Proof. (i)

$$\begin{aligned}
\mathscr{E}V_i &= \sum_{j=T_i+1}^{T_{i+1}} \mathscr{E}X_j \\
&= \sum_{\zeta} \zeta^{(1)}\nu(\zeta) \\
&= A \sum_{\zeta} \zeta^{(1)}\beta(\zeta) \\
&= A \sum_{\xi} \xi\alpha(\xi).
\end{aligned}$$

(ii)

$$\begin{aligned}
\mathscr{E}\| V_i \|^2 &\leqq \sum_{k=1}^{\infty} a_k(kB)^2 \\
&\leqq B^2 \sum_{k=1}^{\infty} k^2\lambda^k \\
&= B^2\lambda(1+\lambda)/(1-\lambda)^3.
\end{aligned}$$

Let ρ be a possible value of W_k, and let $Q_1, Q_2, \cdots$, be the finite or infinite subset of $\{T_i\}$ for which $W_k = \rho$. Then since $W_k = \rho$ is equivalent to $S_{T_{k+1}} - S_{T_1} = \rho$, $Z_{T_{k+1}} = \zeta_0$, we have the result

(5) $\Pr\{S_l = \sigma \,|\, \zeta'; Q_i < l \leqq Q_{i+1}\}$ is independent of i.

Suppose now that the initial state is chosen arbitrarily, $Z_m = \zeta'$, i.e. the first m steps $X_1, \cdots, X_m$ are specified. A value σ (or a pair σ, ζ), will be called a possible value (or pair), given ζ', if respectively

$$\Pr\{S_n = \sigma \,|\, Z_m = \zeta'\} > 0,$$

$$\Pr\{S_n = \sigma, Z_n = \zeta \,|\, Z_m = \zeta'\} > 0.$$

THEOREM 1. *Under assumptions* (3) *and* (E), *for* $d = 1, 2$;

(i) *if* $\sum_\xi \xi\alpha(\xi) = 0$, *every point* σ *that is possible given* ζ' *is also recurrent given* ζ',

(ii) *if* $\sum_\xi \xi\alpha(\xi) \neq 0$, *no point is recurrent.*

Proof. (i) Consider the sequence W_k defined in (4), taking $\zeta_0 = \zeta'$. By Lemma 1, (i) it satisfies the conditions for the existence of a recurrent set, given in [1,4]. Let us choose a point ρ belonging to this set and suppose that σ is a point, possible for S_n given ζ', not belonging to it. Then

$$\Pr\{S_l = \sigma \,|\, \zeta'; Q_i < l \leqq Q_{i+1}\} > 0$$

for some i. But by (5) this holds for all i, and hence σ is recurrent.

The set of possible values given ζ' will in general consist of a subset of the cosets of a subgroup of the additive group of one or two dimensional integers respectively.

(ii) The points which are possible for W_k given ζ' are transient for W_k given ζ'. Let ρ be such a point. Now suppose that some point σ is recurrent for S_n. Then by the finiteness of C, some pair (σ, ζ'') must be recurrent. Let $T_1', T_2', \cdots$, be the values of n for which $S_n = \sigma, Z_n = \zeta''$. Now $W_k = \rho$ for some k if and only if $S_l = \rho, Z_l = \zeta'$ for some l. Since ρ is possible for W_k we have

$$\Pr\{S_l = \rho, Z_l = \zeta' | \zeta'; T_i' < l < T_{i+1}'] > 0$$

for some i, and hence for all i since $(S_n = \sigma,\ Z_n = \zeta'')$ is a regenerative event in the sense of (5). This implies the recurrence of ρ and the contradiction establishes the result. If the symmetry condition

$$\begin{aligned} \Pr\{X_n = \xi_n | X_{n-1} = \xi_{n-1}, \cdots, X_{n-m} = \xi_{n-m}\} &= \\ \Pr\{X_n = -\xi_n | X_{n-1} = -\xi_{n-1}, \cdots, X_{n-m} = -\xi_{n-m}\}, \quad n > m&, \end{aligned} \tag{6}$$

is imposed on the transition probabilities, it is easy to see that the stationary distribution on M must be symmetric, $\beta(\zeta) = \beta(-\zeta)$, hence $\alpha(\xi) = \alpha(-\xi)$ and condition (i) of Theorem 1 is satisfied. It can readily be seen that (6) is satisfied by the correlated random walk studied by Gillis in [5], mentioned earlier, and also by the two dimensional process which he discussed in [6]. In both these cases all points are recurrent whatever the initial step.

THEOREM 2. *For $d \geqq 3$, every point is a transient point of S_n.*

Proof. Suppose there exists a point σ such that $\Pr(S_n = \sigma$ for an infinity of $n) = \pi > 0$. Then since the state space C is finite there must exist a ζ_0 such that $S_n = \sigma, Z_n = \zeta_0$ for an infinity of n, with probability π. Using this ζ_0, define a sequence V_i as in (4). Then we have a sequence of independent random variables, for $d \geqq 3$, for which it is not true that every point is transient, which contradicts the assertions of [1, 4].

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APOLOGY

The Editors apologize to the author of the article: "On the mean length of the chords of a closed curve", which appeared in Vol. 4, No. 1, for printing his name incorrectly. The author's name should read Gábor LÜKŐ.